Plan

(1) Free Green funcorory

(2) Photon propagation in a semi-conductor $\Rightarrow$ dielectric function & Feynman diagrams

Last time:

Wick's theorem: decompose expectation values of operator strings into Green functions

Remaining challenges:

(a) figure out what the free Green's functions are

(b) develop a method for systematically organizing the perturbation theory

(c) compute something — photon propagator in matter $\Rightarrow$ dielectric function

(1) Green function of the free electron:

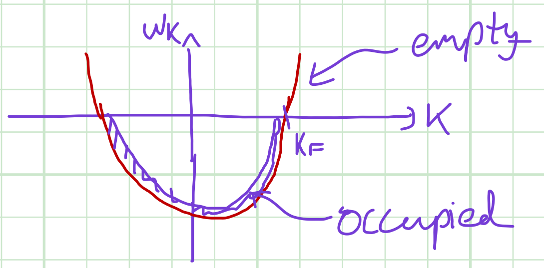
Free $\bar{e}$ Hamiltonian:

$$\mathcal{H}_0 = \sum_K \omega_K \, c_K^\dagger c_K$$

① $\omega_K = \frac{K^2}{2m} - \mu$ for electrons

Ground state of the Fermi sea $[T=0]$:

Plot the dispersion



Consider the K-th mode:

if $\omega_K > 0$ $[|K| > K_F]$ $\Rightarrow n_F(K) = \lim_{\beta\to\infty}\left[\frac{1}{e^{\beta\omega_K}+1}\right] = 0$

if $\omega_K < 0$ $[|K| < K_F]$ $\Rightarrow n_F(K) = 1$ [fill in the plot]

$$n_K(\omega_K) = \begin{cases} 1, & \omega_K > 0 \\ 0, & \omega_K < 0 \end{cases}$$

Hence:

if $K < K_F \Rightarrow$ K-th mode occupied $|Vac\rangle_K = c_K^\dagger |0\rangle$

if $K > K_F \Rightarrow$ mode empty $|Vac\rangle_K = |0\rangle$

Expectation values for Schrodinger operators

$$\langle c_k c_k^+\rangle = {}_k\langle vac|\, c_k c_k^+ |vac\rangle_k = \begin{cases} \text{if } (|k|<k_F) \Rightarrow \overbrace{\langle 0|c_k}^{{}_k\langle vac|}\, c_k c_k^+ \overbrace{c_k^+|0\rangle}^{|vac\rangle_k} = 0 \\ \text{if } (|k|>k_F) \Rightarrow \underbrace{\langle 0|}_{{}_k\langle vac|}\, c_k c_k^+ \underbrace{|0\rangle}_{|vac\rangle_k} = 1 \end{cases}$$

$$= \theta(k-k_F) = (1-n_F(\xi_k))$$

$$\langle c_k^+ c_k\rangle = \theta(k_F-k) = n_F(\xi_k)$$

Hence $G^{(0)}(k,t-t') = -i\langle T\, \hat{c}_k(t)\, \hat{c}_k^+(t')\rangle$ interaction picture since we want $G^{(0)}$

$$= -i\left(\theta(t-t')\,\langle \hat{c}_k(t)\,\hat{c}_k^+(t')\rangle - \theta(t'-t)\langle \hat{c}_k^+(t')\hat{c}_k(t)\rangle\right)$$

To unwrap the expression for the Green's function, we need one more piece of information — how the interaction picture operators time evolve

$$i\partial_t\, \hat{c}(t) = i\partial_t\, e^{+iH_0 t}\, c\, e^{-iH_0 t} = i\left[\, iH_0\, e^{iH_0 t} c\, e^{-iH_0 t} + e^{iH_0 t} c\,(-iH_0)\, e^{-iH_0 t}\right]$$

$$= i\left(+iH_0\, \hat{c}(t) - \hat{c}(t)\, iH_0\right) = [\hat{c}(t), H_0]$$ $\Leftarrow$ we get Heisenberg's EOM

makes sense since with no $\hat{U}$, Heisenberg + inter. pictures are identical.

using the above H_0, we obtain

$$i\partial_t\, \hat{c}_k(t) = [e^{iH_0 t} c_k e^{-iH_0 t},\ w_k c_k^+ c_k]$$

$$= e^{iH_0 t}\,[c_k,\ w_k c_k^+ c_k]\, e^{-iH_0 t}$$

$$= e^{iH_0 t}\, w_k c_k\, e^{-iH_0 t} = w\, \hat{c}_k(t)$$

Hence: $\hat{c}_k(t) = e^{-i\omega_k t} c_k$

Following similar logic for creation operator, we obtain:

$$\hat{c}_k^+(t) = e^{i\omega_k t} c_k^+$$

Using these solutions: $\langle \hat{c}_k(t)\, \hat{c}_k^+(t')\rangle \to e^{i\omega_k(t'-t)} \langle c_k c_k^+\rangle = e^{-i\omega_k(t-t')}\,\theta(\omega_k)$

$\langle \hat{c}_k^+(t')\, c_k(t)\rangle \to e^{i\omega_k(t'-t)} \langle c_k^+ c_k\rangle = e^{-i\omega_k(t-t')}\,\theta(-\omega_k)$

Using these, we obtain: $G^{(0)}(k,t-t') = -i\left[\theta(t-t')\,\theta(\omega_k) - \theta(t'-t)\,\theta(-\omega_k)\right] e^{-i\omega_k(t-t')}$

Similar considerations give us the phonon and photon Green's function (see D.S. book for full details).

$\Rightarrow$ There is one important difference $\Rightarrow$ No "Bose sea" $\rightarrow$ ground state of non-interacting Bosons is the empty state $|vac\rangle_k = |0\rangle_k$

$$G_k^{\text{phonon}(0)}(t) = -i e^{-i\omega_k (t)} \theta(t)$$

only one θ function, no occupied phonon states in equilibrium at $T=0$.

$\omega_k = v_s |k| \Leftarrow$ acoustic

$\omega_k = \omega_0 \Leftarrow$ optical

The photon Green's function is more complicated due to presence of polarizations. We will ignore polarizations for the most part and write:

$$G_{k,\mu}^{\text{photon}(0)}(t) = -i e^{-i\omega_k (t)} \theta(t)$$

$c|k|$

polarization if needed ...

Fourier transforming to get frequency space Green functions:

$$G_k^{\text{phonon}(0)}(\omega) = -i \int_{-\infty}^{\infty} dt \; e^{i\omega t} e^{-i\omega_k t} \theta(t)$$

$$= \lim_{\epsilon \to 0^+} -i \int_0^{\infty} dt \; e^{i(\omega-\omega_k)t} e^{-\epsilon t}$$

needed for convergence

$$= \lim_{\epsilon \to 0^+} \frac{1}{\omega - \omega_k + i\epsilon}$$

Following same procedure for electrons, we get:

$$G_k^{el}(\omega) = \frac{1}{\omega - \omega_k + i\delta_k} \qquad \delta_k = \delta \, \text{sign}(\omega_k)$$

(2) Photon propagator + the dielectric function:

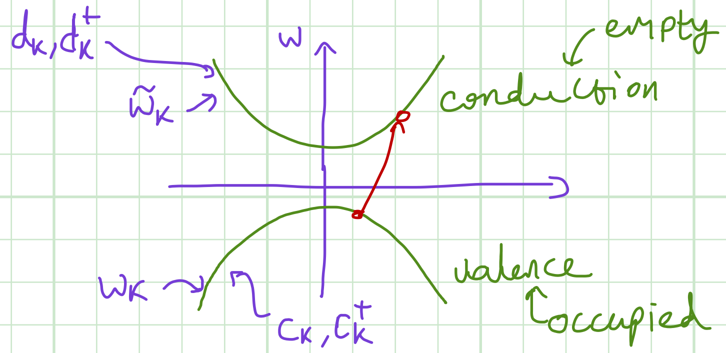
consider the propagation of a photon through a semi-conductor

What is Hamiltonian:

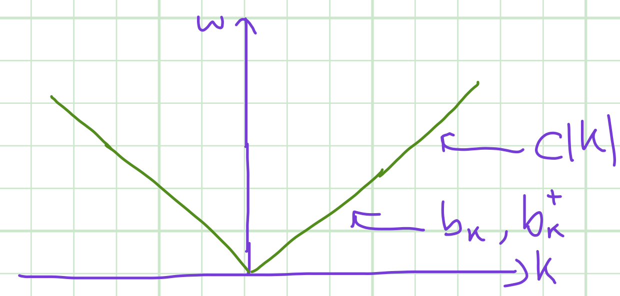
$$H = \underbrace{H_{el} + H_{photon}}_{H_0} + V_{el\text{-}photon}$$

The "free" part

Electrons:



photons:



$$H_0 = \sum_k w_k c_k^\dagger c_k + \tilde{\omega}_k d_k^\dagger d_k + c|k|\, b_k^\dagger b_k$$

Electron Photon interactions:

$$V_{int} = -q\int X(r)E(r)\,dr$$

Electric field

e^- displacement

e^- charge

[Expression from Lecture 6]

$$= -q\int đk\; X(k)\,E(-k) \quad \longleftarrow \quad đk = \frac{d^dk}{(2\pi)^d}$$

$$= -q\int đk\; X(k)\sqrt{\frac{\hbar c|k|}{2\varepsilon V}}\left[a_{-k} - a_k^\dagger\right]$$

Displacement field for electrons:

- Remember that current $j = \dot{x} \Rightarrow j(k,\omega) = \omega X(k,\omega)$
- Electron current operator

Single Band Picture:

$$\dot{\rho}(r) = \nabla\cdot J(r)$$

$$\Downarrow$$

$$i\hbar\,\partial_t \hat{c}_r^\dagger(t)\,\hat{c}_r(t) = -\frac{\hbar}{2m}\left(\left[\nabla^2 \hat{c}_r^\dagger(t)\right]\hat{c}_r(t) - \hat{c}_r^\dagger(t)\left[\nabla^2 c_r(t)\right]\right)$$

$$= -\frac{\hbar}{2m}\nabla\cdot\left(\left[\nabla \hat{c}_r^\dagger(t)\right]\hat{c}_r(t) - \hat{c}_r^\dagger(t)\left[\nabla \hat{c}_r(t)\right]\right)$$

$$\Rightarrow J_\mu(r,t) = \frac{i\hbar}{2m}\left(\left[\nabla_\mu \hat{c}_r^\dagger(t)\right]\hat{c}_r(t) - \hat{c}_r^\dagger(t)\left[\nabla_\mu \hat{c}_r(t)\right]\right)$$

$$\Rightarrow J_\mu(k,t) = \sum_p \frac{\hbar}{m}\left(p + k/2\right)\hat{c}_p^\dagger\,\hat{c}_{p+k}$$

Our case is slightly more complicated because we have two bands $\Rightarrow$ interband current operator:

$$\hat{J}_{cv}(k,t) = \sum_p \left[\tilde{M}_{p,k}\, \hat{c}_p^\dagger \hat{d}_{p+k} + h.c.\right]$$

$\Rightarrow$ This is the important current for polarization, because no $c \to c$ or $v \to v$ transitions in ground state.

Plugging this into V_{int}

$$V_{int} = \int dp\, dk\, \underbrace{M_{p,k}}_{\text{captures all the ME information.}} \left[\hat{c}_p^\dagger \hat{d}_{p+k} + \hat{d}^\dagger_{p+k} c_p\right] \left(a_{-k} - a_k^\dagger\right)$$

Let us now compute the photon Green's function.

$$G_k^{photon}(t,t') = -i\langle T\, a_k(t)\, a_k^\dagger(t')\rangle$$

$$= \sum_{n=0}^{\infty} \frac{(-i)^{n+1}}{n!} \int_{-\infty}^{\infty} dt_1 \dots dt_n\, \frac{{}_0\langle T\, \hat{a}_k(t)\, \hat{V}(t_1)\, \hat{V}(t_2) \dots \hat{V}(t_n)\, \hat{a}_k^\dagger(t')\rangle_0}{{}_0\langle S(\infty,-\infty)\rangle_0}$$

Let's concentrate on numerator first:

$n=0$: $-i\langle T\, \hat{a}_k(t)\, \hat{a}^\dagger(t')\rangle = -i e^{-ic|k|(t-t')}\, \Theta(t-t')$

$n=1$: $(-i)^2 \int_{-\infty}^{\infty} dt_1\, {}_0\langle T\, \hat{a}_k(t) \int dp\, dq\, M_{p,q} \left(\hat{c}_p^\dagger(t_1)\, \hat{d}_{p+q}(t_1) + \hat{d}^\dagger_{p+q}(t_1)\, \hat{c}_p(t_1)\right)\left(\hat{a}_{-q}(t_1) - \hat{a}_q^\dagger(t_1)\right)$
$\hat{a}_k^\dagger(t')\rangle_0$

Apply Wick's theorem $\Rightarrow$ separate the types of particles

$$-1\int_{-\infty}^{\infty} dt_1 \int dp\, dq\, M_{p,q}\, {}_0\langle T\, \hat{a}_k(t)\left[\hat{a}_{-q}(t_1) - \hat{a}_q^\dagger(t_1)\right] \hat{a}_k^\dagger(t')\rangle_0 \Big[{}_0\langle \hat{c}_p^\dagger(t_1)\rangle_0\, {}_0\langle \hat{d}_{p+q}(t_1)\rangle_0 + {}_0\langle \hat{d}^\dagger_{p+q}(t_1)\rangle_0\, {}_0\langle \hat{c}_p(t_1)\rangle_0\Big]$$

3-photon term is zero

both zero

$\Rightarrow$ No linear term!

$n=2$: $(-i)^3 \int_{-\infty}^{\infty} dt_1\, dt_2 \int dp_1\, dq_1\, dp_2\, dq_2\, {}_0\langle T\, \hat{a}_k(t)\left[\hat{a}_{-q_1}(t_1) - \hat{a}^\dagger_{q_1}(t_1)\right]\left[\hat{a}_{-q_2}(t_2) - \hat{a}^\dagger_{q_2}(t_2)\right] \hat{a}_k^\dagger(t')\rangle_0$

${}_0\langle T \left(\hat{c}^\dagger_{p_1}(t_1)\, d_{p_1+q_1}(t_1) + \hat{d}^\dagger_{p_1+q_1}(t_1)\, c_{p_1}(t_1)\right)$

$\times \left(\hat{c}^\dagger_{p_2}(t_2)\, d_{p_2+q_2}(t_2) + \hat{d}^\dagger_{p_2+q_2}(t_2)\, c_{p_2}(t_2)\right)\rangle_0$

Two ways to pair up the Bosons

$$-\langle T\,\hat a_k(t)\,\hat a^+_{q_1}(t_1)\rangle_0\,\langle T\,\hat a_{-q_2}(t_2)\,\hat a^+_k(t')\rangle$$
$$=-\left[-i\,G^{ph}_k(t-t_1)\,\delta(k-q_1)\right]\left[-i\,G^{ph}_k(t_2-t')\,\delta(k+q_2)\right]$$

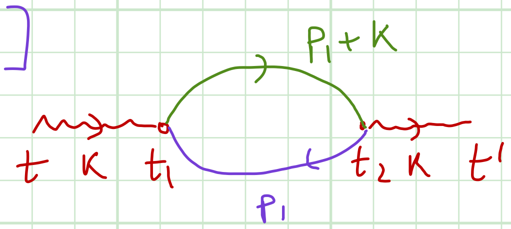
$$-\langle T\,\hat a_k(t)\,\hat a^+_{q_2}(t_2)\rangle_0\,\langle T\,\hat a_{-q_1}(t_1)\,\hat a^+_k(t')\rangle$$
$$=-\left[-i\,G^{ph}_k(t-t_2)\,\delta(k-q_2)\right]\left[-i\,G^{ph}_k(t_1-t')\,\delta(k+q_1)\right]$$

Two ways to pair up the fermions:

$$\langle T\,(\hat c^+_{p_1}(t_1)\,d_{p_1+q_1}(t_1)\,d^+_{p_2+q_2}(t_2)\,c_{p_2}(t_2))\rangle_0+\langle T(d^+_{p_1+q_1}(t_1)\,c_{p_1}(t_1))\times(\hat c^+_{p_2}(t_2)\,d_{p_2+q_2}(t_2))\rangle_0$$
$$=\langle T\,c^+_{p_1}(t_1)\,c_{p_2}(t_2)\rangle_0\,\langle T\,d_{p_1+q_1}(t_1)\,d^+_{p_2+q_2}(t_2)\rangle+\langle T\,d^+_{p_1+q_1}(t_1)\,d_{p_2+q_2}(t_2)\rangle_0\,\langle T\,c_{p_1}(t_1)\,c^+_{p_2}(t_2)\rangle_0$$
$$=\left[i\,G_{v,p_1}(t_2-t_1)\,\delta(p_1-p_2)\right]\left[-i\,G_{c,p_1+q_1}(t_1-t_2)\,\delta(p_1+q_1-p_2-q_2)\right]$$
$$+\left[i\,G_{c,p_1+q_1}(t_2-t_1)\,\delta(p_1+q_1-p_2-q_2)\right]\left[-i\,G_{v,p_1}(t_1-t_2)\,\delta(p_1-q_1)\right]$$

Putting everything together, we ~~obtain~~:

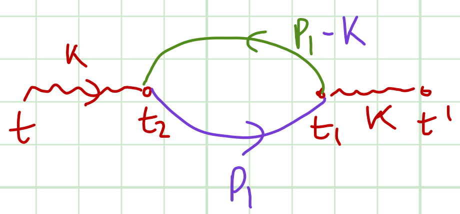
[1,1]



$$\overset{1}{\delta(k-q_1)}\;\overset{2}{\delta(k+q_2)}\;\overset{3}{\delta(p_1+q_1-p_2-q_2)}\;\overset{4}{\delta(p_1-p_2)}$$

$1 \Rightarrow q_1=k \quad \Rightarrow p_1+q_1=p_1+k$

[2,1]



$$\delta(k-q_2)\,\delta(k+q_1)$$

$q_1=-k \Rightarrow p_1+q_1=p_1-k$

etc.